Title:	**Double Pendulum: A Bridge between Regular Dynamics and Chaos**
Original: **Revision:**	July 26, 2005 (v1) Nov 14, 2005 (v2), May 23, 2006 (v3), June 22, 2009 (v4)
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Appropriate Level:	Regents Physics
Abstract:	Of all physical phenomena, the simple pendulum is perhaps the best suited to introduce students to the concept that the natural world can be described in a mathematical language and provides an entry point into conceptual, analytic and experimental techniques. The double pendulum is a system that behaves exactly like the simple pendulum for small amplitudes but is chaotic for larger amplitudes providing students with an introduction to the fascinating ideas about chaos theory while tying it closely to concepts and techniques taught at the Regents Physics level.
Time Required:	Two to three 40-minute class periods.
NY Standards Met:	Standard 1: M1.1, M2.1, M3, S1, S2, S3 Standard 6: 2.2, 2.3, 2.4, 5.1, 5.2 Standard 4: 4.1 (i, ii, iv) Mathematical analysis, scientific inquiry, modeling, plotting, factors affecting pendulum motion. Hypothesis testing and verification.
Special Notes:	"**Chaotic Double Pendulum**" is a kit available through the CIPT equipment lending library. Created by the CNS Institute for Physics Teachers via the Nanoscale Science and Engineering Initiative under NSF Award # EEC-0117770, 0646547 and the NYS Office of Science, Technology & Academic Research under NYSTAR Contract # C020071

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Cognitive Objectives:
By the end of the lab, students should:

- Appreciate that models work within specified limits.
- Recognize that there are qualitative differences between chaotic and regular dynamics, including predictability and variance of measured quantities.
- Understand that chaotic systems are sensitive to initial conditions.
- Have improved their hypothesis creation and testing skills.

Teacher preparation time:
To set up the module should take less than 30 minutes.

Materials:

- Double pendulum kits from CIPT lending library, consisting of double pendulums constructed from in-line skate wheels. These will be referred to as the "student double pendulums." These pendulums require very stable ring stands for support and come with washers to add weight.
- Machined double pendulum from the CIPT lending library, which will be referred to as the "demonstration double pendulum".
- Rubber bands and tongue depressors to mechanically restrict the demonstration double pendulum to simple pendulum motion (optional).
- Protractors
- Photogates
- Strobe light with variable flash rate setting.
- Stopwatches and rulers (to be made available as needed for free inquiry work)

Assumed Prior Knowledge of the Students:

- Basic information regarding simple pendulums.
- Graphing skills (e.g., interpreting velocity vs. time graphs)
- The role of hypothesis testing in scientific inquiry.
- Experience using photogates..

Tips for the Teacher:
Here is a suggested teaching strategy for presenting this lesson, based on the 5-E instructional model for inquiry-based learning of science:

1. Free Exploration
    - Give students a very short introduction to the equipment. Make sure they are aware that each setup consists of two side-by-side double pendulums. If a group will be using the demonstration double pendulum, advise them to take care not to pinch their fingers in the apparatus and to stop the oscillations by steadying the very top of the upper arm. Students using the student double pendulums

should be shown how to steady the apparatus during use and to use the two nuts for attaching washers. The washers are used to add weight to the pendulums by being placed between the nuts.
- Give students a short period of time to freely explore the behavior of the double pendulums and to jot down some observations in their student lab reports.
- Bring the students back as a group to share their observations. Explore ways in which the double pendulum motion can be described mathematically (e.g., period, angle of release, number and direction of flips). Connect their observations to what students already know about simple pendulums. If desired, make the lower joint of the demonstration double pendulum immobile with tongue depressors and rubber bands to reinforce student knowledge of simple pendulum phenomena.

2. Sensitivity of Double Pendulum Motion to Initial Conditions
   - Challenge students to release two side-by-side double pendulums with initial conditions as identical as possible. For low angles of release, this should result in simple, periodic motion (non-chaotic) and the side-by-side pendulum motions should remain nearly identical. For higher angles of release, the two pendulums will quickly diverge.
   - Have them read the Introduction to the Student Section (which can also be read ahead as a pre-lab activity). After reading the Introduction, emphasize that chaotic motion is not completely random; Newton's laws of motion still apply. Rather, the application of these laws quickly becomes too complex for us to predict exactly what will happen. In other words, sensitivity to initial conditions quickly results in chaotic, aperiodic behavior.
   - If a variable strobe light is available, demonstrate in a darkroom that non-chaotic pendulum motion can be made to appear to sit still when the flash rate is in synchrony with the pendulum's period. However, when the pendulum is in the chaotic mode, it is impossible to make it appear to be sitting still because there is no constant period to the motion.

3. Determination of a 'Critical Angle'
   - Following the protocol in the Student Section, students should explore the conditions under which chaotic (non-linear) motion can occur. Make sure students understand how to use the protractor to determine the angles of release θ_1 and θ_2 (see diagram in student lab report). For the student double pendulums, the critical angle is about 25° without any washers attached, but higher if washers are attached. For the demonstration double pendulum, the critical angle is about 100°.

Suggested values for students using the demonstration double pendulum:

θ_1	θ_2	Chaotic or Periodic?

80	80	
90	90	
100	100	
110	110	
120	120	

(Results will vary.)

Suggested values for students using the student double pendulums:

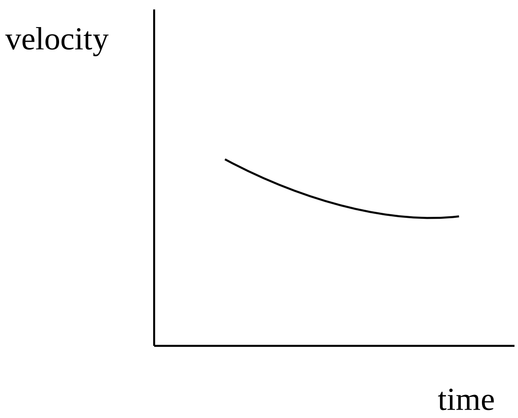
θ_1	θ_2	Chaotic or Periodic?
15	15	
20	20	
25	25	
30	30	
35	35	

(Results will vary.)

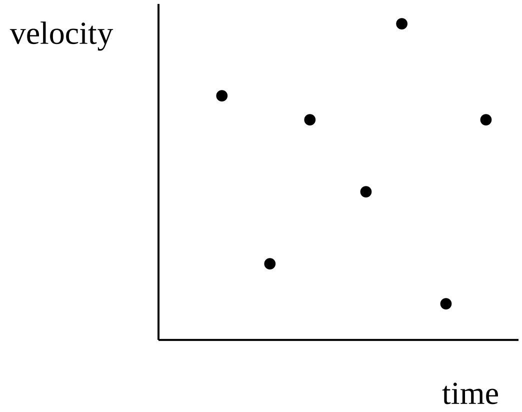
- After students have collected the data, bring them back together to discuss their results. Brainstorm the reasons for any discrepancies amongst the groups' results. Part of this discussion may center on how to recognize motion as chaotic.

4. Making it Quantitative
   - Using just one double pendulum and a photogate, measure how the velocity of the double pendulum's lower arm varies over time. Set the software to "One Gate" mode. Depending on the number of photogates available, these measurements can be made independently or as a demonstration.
   - The resulting graph of velocity vs. time for chaotic motion should have a high variance. For low angles of release (linear mode), the graph will be quite regular and significantly different from the chaotic mode. As the double pendulum loses energy, the transition from chaotic to periodic motion should be graphically dramatic. Have students sketch the results on the axes provided in the student lab report. The graph for the non-chaotic mode should be relatively constant, with a declining velocity as energy is lost to friction. The graph for the chaotic mode should vary enormously, with velocities ranging from very large to very small.

Typical Results:



Non-Chaotic Mode



Chaotic Mode

- If additional mathematical work is desired, students can be instructed to make a histogram of the velocity vs. time data. The spread in the data for the non-chaotic motion will be markedly narrow in comparison to spread for the chaotic mode data.

5. Inquiry

- As the students play with the pendulums, they will undoubtedly discover a number of interesting phenomena. These can be used as an opportunity for hypothesis creation and experimental testing of hypotheses. Here are some examples of the sort of hypotheses that might be of interest:
  - Does the total number of flips depend on the height of release?
  - Does the direction of the flips (clockwise vs. counter-clockwise) depend on which way the pendulum initially falls? Does this change if the observation time is extended?
  - Can human imprecision be eliminated by somehow ‘mechanically’ releasing the two double pendulums such that chaotic behavior does not result, even at large angles of release?
  - Is there a limit to the number of consecutive flips?
  - Does the behavior of one double pendulum affect the motion of the other? In other words, if we release one while keeping the other stationary, will they behave the same when are both are swinging at the same time?
  - If the pendulum is to the ‘right’ at a certain time, is it more likely to be in that direction again 5 seconds later? How about 10 seconds later?
  - Does the number of washers affect the number of flips or the amount time it takes for the pendulum to settle down into a periodic mode?

- To structure student work on their inquiry topic, use the format provided in the student section.
- Review each group's work before they begin the data-gathering phase of their research.
- If desired, have students share their studies with the other groups with a short oral report.

Background Information for the Teacher:

Of all physical phenomena, the simple pendulum is perhaps the best suited to introduce students to the concept that the natural world can be described in a mathematical language. Conceptually, the system can serve as an introduction to energy conservation, free-body diagrams forces and vectors. Analytically, the single pendulum is ideally suited to explain the small-angle approximation, obtaining simple differential equations from Newton's second law, and the solution of the simple harmonic oscillator equation either by ansatz or integral calculus. Experimental measurements of periods and velocity provide a venue for data analysis, error estimation and graphical presentation of results.

The purpose of this module is to introduce students to the double pendulum system, which for small angles behaves exactly like the simple pendulum, but for large amplitudes is chaotic. The module expands on the concepts, analysis and measurements that are usually performed on the simple pendulum to provide an introduction to chaos theory allowing students to make hypothesis and then test them experimentally.

Background on Chaos (adapted from Nonlinear Dynamics and Chaos, with applications to Physics, Biology, Chemistry and Engineering by Steven Strogatz, Perseus Books 1994)

The subject of dynamics began in the mid-1600s when Newton invented differential equations, discovered his laws of motion and universal gravitation, and combined them to explain Kepler's laws of planetary motion. Specifically, Newton solved the two-body problem (motion of the earth around the sun). Subsequent generations of mathematicians and physicists tried to extend Newton's methods to the three-body problem and after decades of effort it was eventually realized that it was impossible to solve, in the sense of obtaining explicit formulas for the motion of three bodies. At this point the situation seemed hopeless. The breakthrough came with the work of Poincare in the late 1800s. He introduced a new point of view that emphasized qualitative rather than quantitative questions. For example, instead of asking for the exact positions of the planets at all times, he asked "Is the solar system stable forever, of will some planets eventually fly off to infinity". Poincare developed a powerful geometric approach to analyzing such questions and he was also the first person to glimpse the possibility of chaos, in which a deterministic system exhibits aperiodic behavior that depends sensitively on the initial conditions, thereby rendering long-term prediction impossible.

But chaos remained in the background for the first half of this century; instead dynamics was largely concerned with non-linear oscillators and their applications to technologies such as the radio, radar, phase-locked loops and lasers. It is the work of Lorenz in 1963 that showed that solutions to some equations used in simple weather models never settled down to equilibrium or to a periodic state, and if he started his simulations from two slightly different initial conditions, the resulting behavior would soon become totally different. The implication was that the system was inherently unpredictable – tiny errors in measuring the current state of the atmosphere (or any other chaotic system) would be amplified rapidly making it impossible to forecast. But Lorenz also showed that there was structure in chaos, and the work of Feigenbaum in the 1970s showed there are certain laws governing the transition from regular to chaotic behavior; roughly speaking, completely different systems can go chaotic in the same way. By the 1980s, chaos had developed into a large field with applications in physics, biology, chemistry and engineering. For further reading, the book cited above by Strogatz is an excellent resource.

Motivation behind this module

In recent discussions about the teaching of physics the "two-brain syndrome" has been identified as a significant problem (See Physics and Society, January 2005 and July 2005). The problem is that while students might perform well on tests, they do not internalize the scientific concepts and appreciate the value of scientific inquiry. In the July 2005 Commentary in Physics and Society, David Griffiths writes that the problem is caused because "courses drowned the students in a flood of 'facts', concepts and canned calculational procedures [where] students could hardly be expected to appreciate the beauty and honestly of science". At the heart, physics is about a passion for learning about the unknown, and teaching of physics should not leave the students feeling that all problems have already been solved, but rather kindle a curiosity to understand natural phenomena. The double pendulum is ideally suited to this purpose because for a certain set of initial conditions it behaves exactly like the simple pendulum where students can analytically or experimentally fully understand and predict the motion. However, for other starting conditions, the pendulum becomes chaotic making it impossible to predict the exact trajectory. While the chaotic motion is inherently unpredictable, in the optional take-home section an attempt is made to show how chaos can be understood geometrically and to encourage students to think about the transition from simple textbook problems to the more complex and less understood phenomena in the real world.

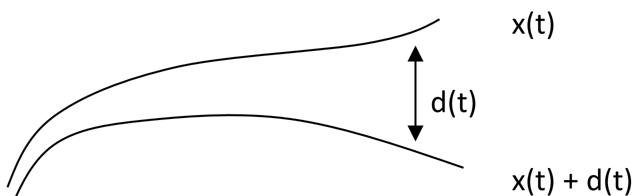
Definition of chaos

There is no universally excepted definition of chaos. But almost everyone would agree on the following ingredients: Chaos is aperiodic long-term behavior in a deterministic system that exhibits sensitive dependence on initial conditions.

In this context, the phrase aperiodic long-term behavior means that the motion does not settle down to a fixed point or a periodic orbit. Since the double pendulum loses energy to the environment, after some time the motion does becomes periodic and it eventually stops at a stationary fixed point. In this sense it is only the theoretical double pendulum without energy losses that would really be a chaotic system.

A deterministic system means that the system has no random or noisy inputs. The irregular behavior is intrinsic and arises from the system's non-linearity rather than from any noisy driving forces.

Sensitive dependence to initial conditions means that nearby trajectories separate exponentially fast, i.e. two identical systems set up together in the same way such that the initial conditions are arbitrarily close together will have their trajectories rapidly diverge. To make this more concrete, consider two trajectories as shown below, where at some time t the trajectories are at position x(t) and x(t) + d(t), then the statement of chaos would be that d(t) ~ d(0) exp [L t] , where the average value of L is called the Lyapunov exponent, and if this is positive it means that the two trajectories are quickly separating from each other.



To keep the module simple, we do not distinguish between **ergodicity** and **chaos**. In this context we use the term ergodicity to imply that the trajectory explores all of the available phase space (for example, if we define the motion in terms of two variables, omega which is the angular velocity, and phi which is the angle of the bottom pendula, then the phase space would be a plot of omega versus phi as seen below).

(Figures taken from www.phy.davidson.edu showing on the left a non-ergodic or regular trajectory in phase space, and on the right a chaotic or ergodic trajectory.)

The interested student can be directed to:

http://www.physics.northwestern.edu/ugrad/vpl/mechanics/pendulum.html
http://scienceworld.wolfram.com/physics/DoublePendulum.html

Equipment List

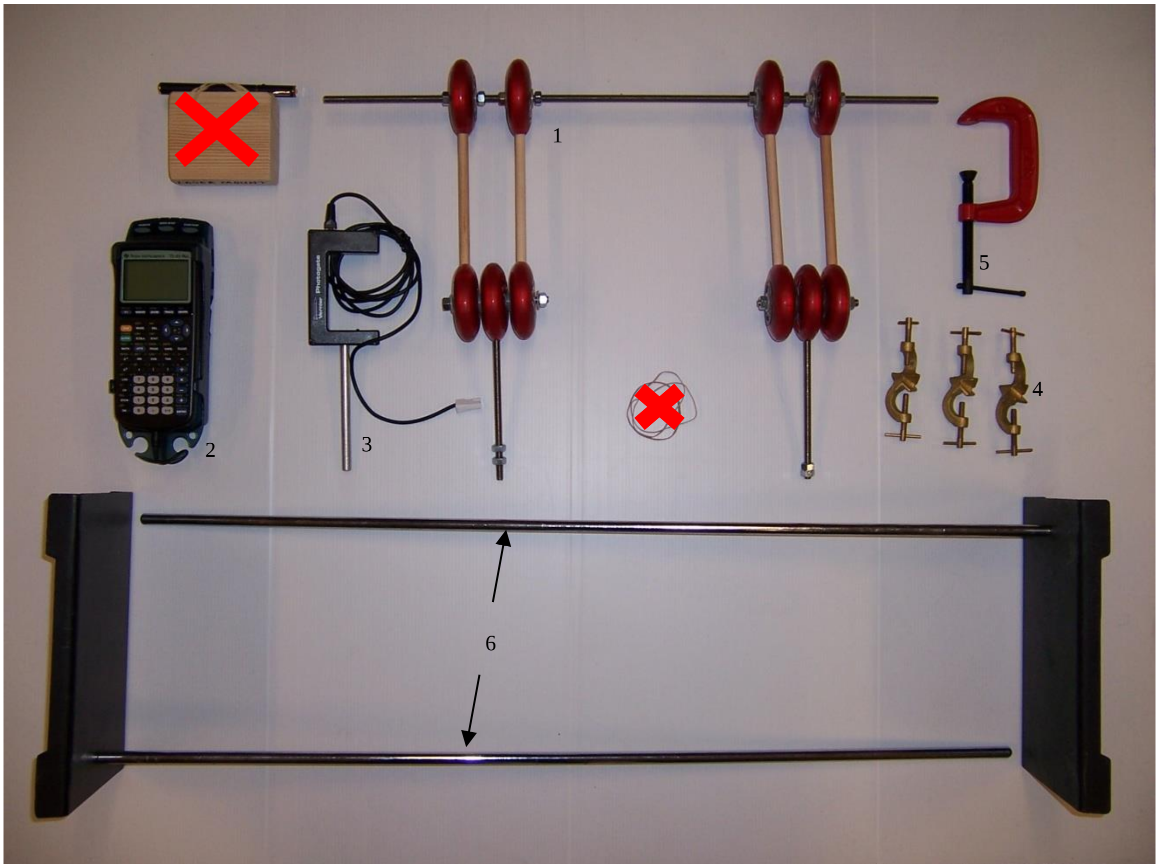


Photo ID	No.	Item
1	1	Double pendulum apparatus
2	1	TI-83 Plus graphing calculator with Vernier Lab Pro Interface
3	1	Vernier Photogate
4	3	Right-angle ring stand clamps
5	1	C-clamp
6	2	Large ring stands
	1	Strobe light – supplied by teacher – not shown
	20	Washers – not shown
	1	Protractor – not shown

CHAOTIC DOUBLE PENDULUM
Student Lab Report

Name ______________________
Date _______________________

Introduction:

In this activity you will experiment with a chaotic system. Chaos is a word used to describe systems that have the following two particular properties:

1. The motion is unpredictable or widely fluctuating.
2. The motion is very sensitive to initial conditions.

Chaos theory tries to make sense out of seemingly random systems such as the weather, turbulence or stock prices. While we may not be able to easily solve the equations for such systems, we can nonetheless develop a language to understand such systems.

One popular example of the sensitivity to initial conditions is the butterfly effect. The butterfly effect is the notion that a butterfly flapping its wings today ever so slightly changes the initial conditions for any equations that you could use to describe the weather. The final outcome of these equations depends critically on initial conditions (i.e. small changes in initial condition causes large changes in predictions.) For example a computer model running both with and without the tiny effect of a butterfly flapping its wings could predict two very different outcomes like hurricanes or droughts. Since we can never know the exact initial condition, this means that chaotic systems are inherently unpredictable over the long term.

Procedure:

Section 1: Free Exploration of the Double Pendulum

Observations:

Question #1: Suggest two characteristics of the double pendulum's chaotic motion that you could use to describe it mathematically.

Question #2: How does the behavior of the double pendulum differ from the behavior of a simple pendulum?

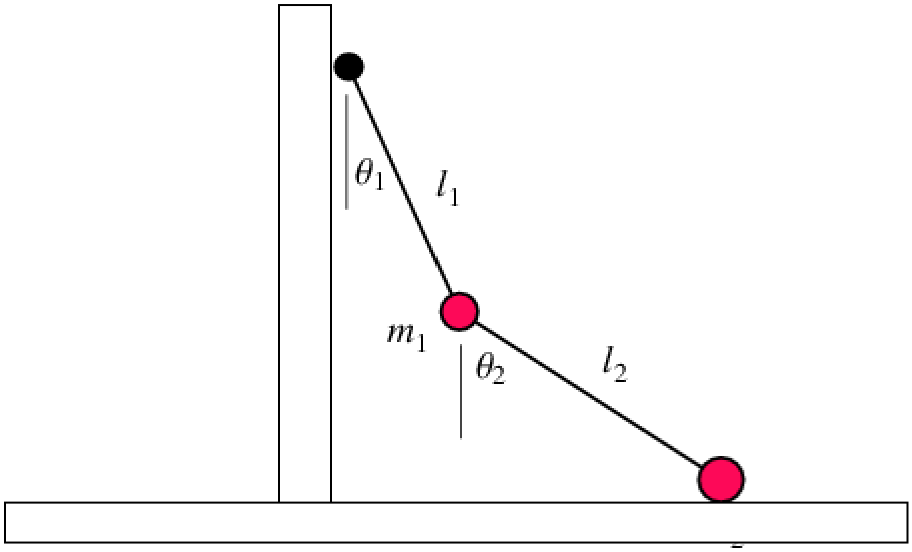
Section 2: Sensitivity of Double Pendulum Motion to Initial Conditions

Question #3: Do double pendulums follow Newton's Laws of motion? Is the motion random?

Question #4: No matter how carefully the two side-by-side double pendulums are released, their motions quickly diverge. Why do they not stay the same?

Question #5: Why is it impossible to adjust the strobe light such that the double pendulum appears motionless when the pendulum is behaving chaotically?

Section 3: Determination of a 'Critical Angle'



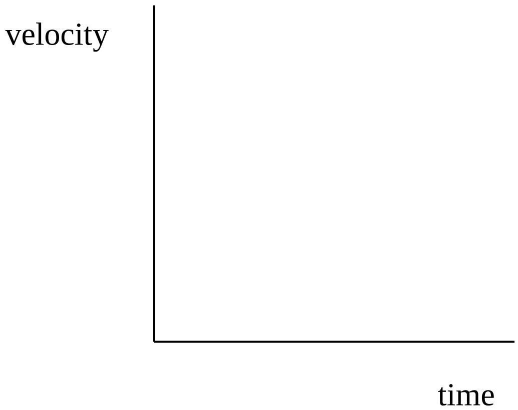
Use the diagram above to indicate the angles of release, θ_1 and θ_2, which are measured with respect to the vertical.

θ_1	θ_2	Chaotic or periodic?

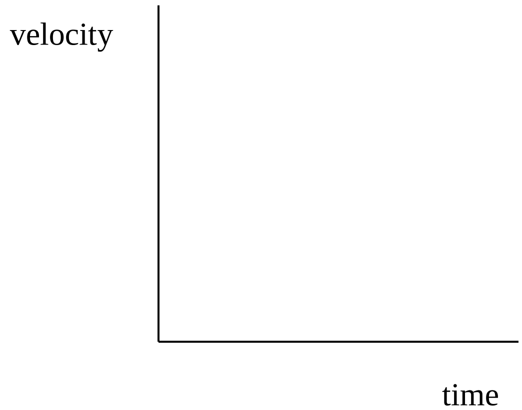
Question #6: Speculate on why not all groups found the same critical angle.

Section #4: Making it Quantitative

Your teacher will show you how to set-up and operate the photogates you will be using to find the velocity of the lower arm of the double pendulum. After collecting the data, sketch the velocity vs. time graphs for this motion when the arm is in the chaotic mode and when it is in the non-chaotic mode.



Non-Chaotic Mode



Chaotic Mode

Question #7: Which mode of oscillation, non-chaotic or chaotic, has a wider variation in velocity? Why?

Section #5: Inquiry

You will now have an opportunity to derive a hypothesis regarding the motion of the double pendulum and to test that hypothesis.

A. Hypothesis

B. Procedure

C. Data

D. Conclusions